Successive Control Barrier Functions for Nonlinear Systems

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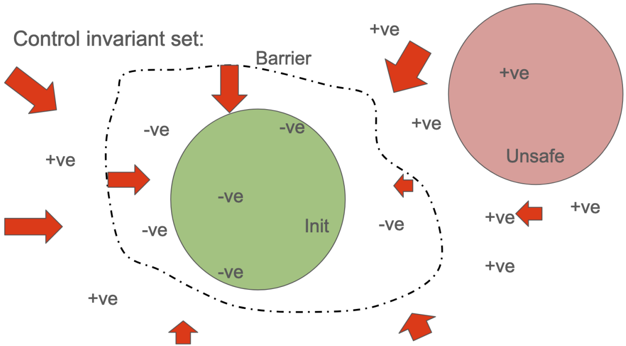
Control Barrier Functions - Exponential [Kong et al.]

- State: $\vec{x} \in \mathbb{R}^n$
- Control inputs: $\vec{u} \in \mathbb{R}^m$
- $\dot{\vec{x}} = f(\vec{x}, \vec{u}),\ X \subseteq \mathbb{R}^n$,



- $B(\vec{x}) > 0$ for all $\vec{x} \in X_u$ (B is **positive** when unsafe)
- $B(\vec{x}) \leq 0$ for all $\vec{x} \in X_i$ (B is **negative** when init)
- for all $\vec{x} \in \mathbb{R}^n$ there **exists a control input** $\vec{u} \in U$ s.t. $\nabla B(\vec{x}) \cdot f(\vec{x}, \vec{u}) \leq -\lambda B(\vec{x})$

Control Barrier Functions - Exponential [Kong et al.]

It's a hard problem:

- Trying to synthesize *Barrier* and *Control* simultaneously
- Bilinearity

- State: $\vec{x} \in \mathbb{R}^n$
- Control inputs: $\vec{u} \in \mathbb{R}^m$
- $\dot{\vec{x}} = f(\vec{x}, \vec{u})$, $X \subseteq \mathbb{R}^n$,

- $B(\vec{x}) > 0$ for all $\vec{x} \in X_u$ (B is **positive** when unsafe)
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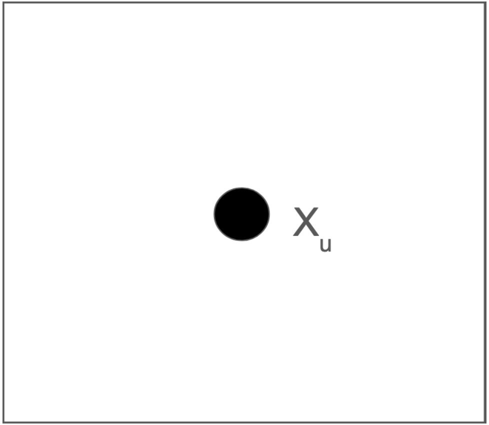
One Barrier is not enough

Computing Barrier:

- Fix $u = u_i \in U$
- Barrier function: $B_i(\vec{x})$
- $\dot{\vec{x}} = f(\vec{x}, u_i)$



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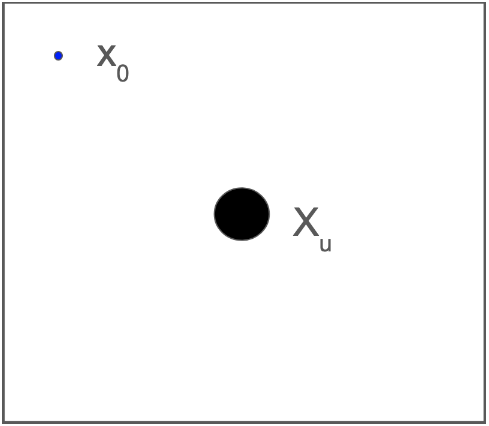

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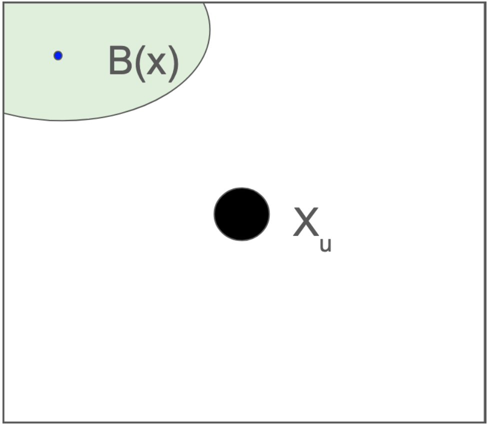

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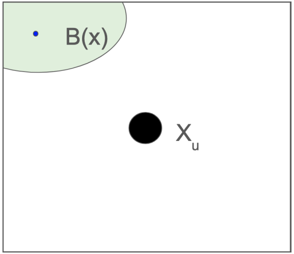

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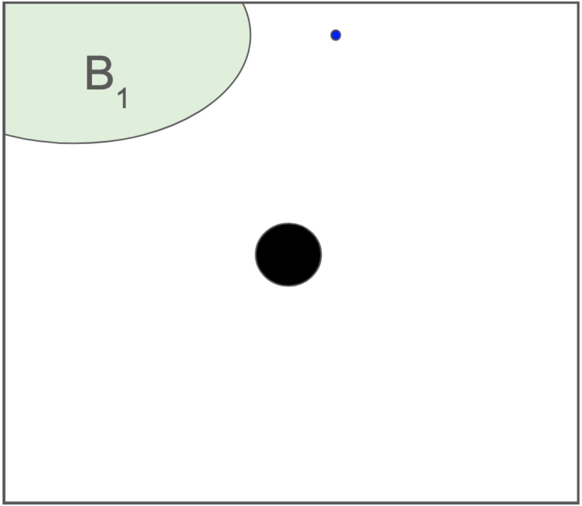
BF $\rightarrow$ CBF ($u = u_i$)

$\rightarrow$ **Easier computation [SOS]**

$\rightarrow$ **Very conservative**

Multiple Barriers



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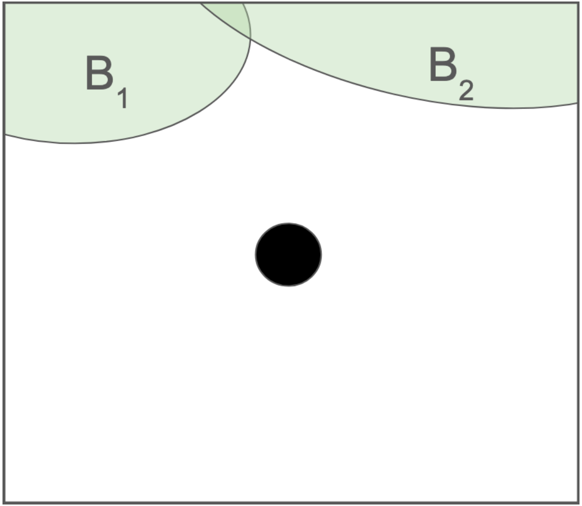

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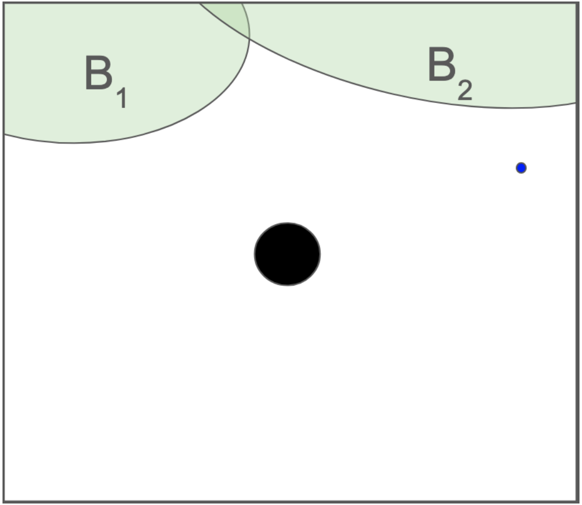

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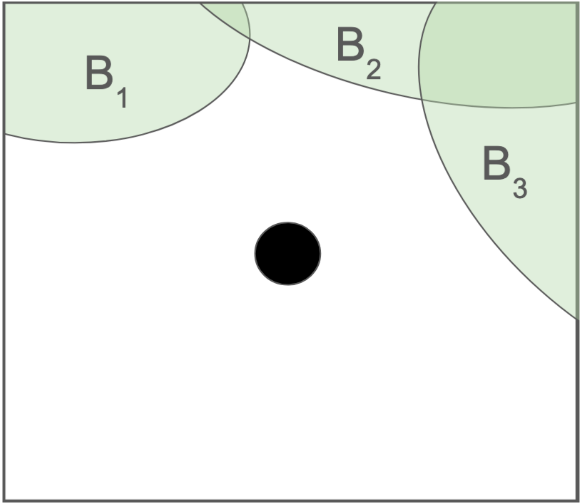
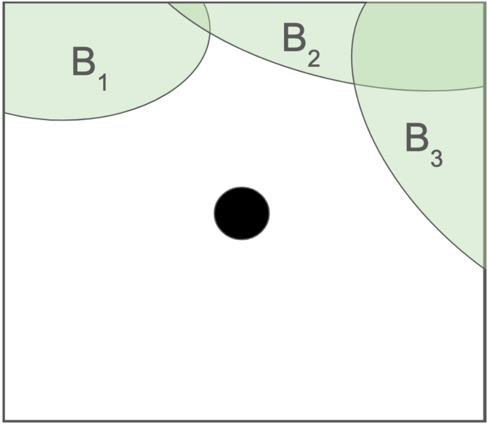


Multiple Barriers

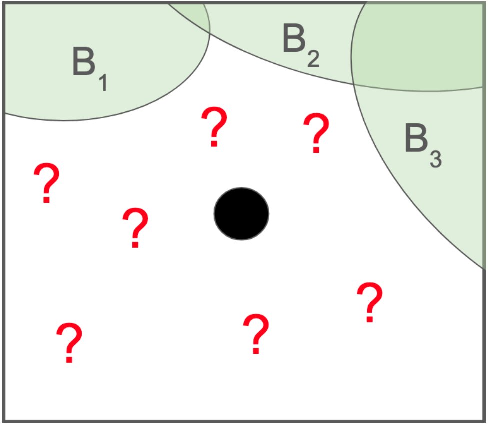
Combining Multiple Barriers:

- $\min(B_1, B_2, B_3) = B^{(1)}$
- Boolean Combination $\rightarrow$ Nonmsooth Analysis [Egerstedt]



Can we do better?

- We have multiple barriers,
- We have a controlled invariant (CI) region,
- Can we add more states to the CI region?



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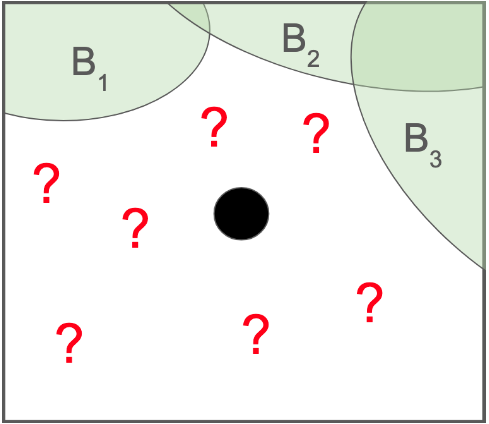
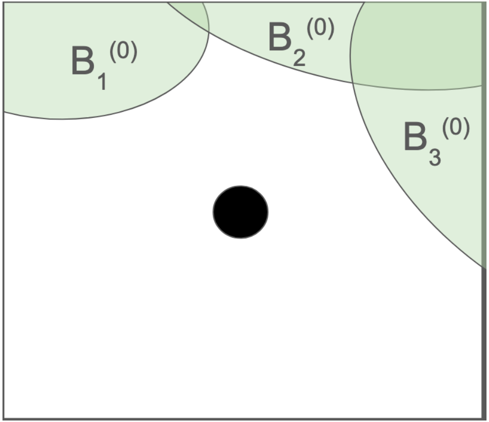


Successive barrier functions

Successive barrier functions

- $\nabla B_i \cdot f(\vec{x}, \vec{u}_i) \leq -\lambda B_i(\vec{x})$,
- Holds $\forall \vec{x} \in \mathbb{R}^n$



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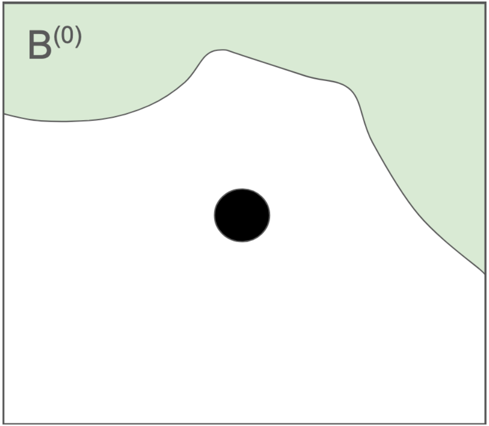
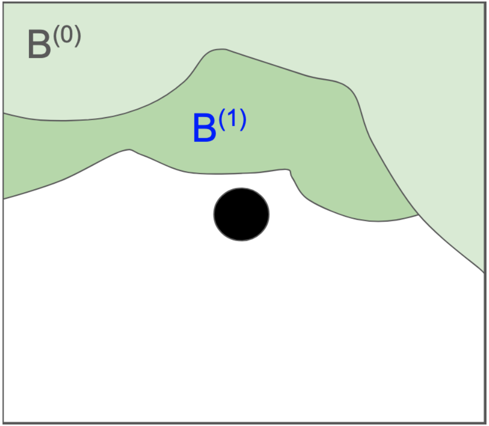


Successive barrier functions

- $\nabla B_i \cdot f(\vec{x}, \vec{u_i}) \leq -\lambda B_i(\vec{x})$,
- ~~Holds $\forall \vec{x} \in \mathbb{R}^n$~~
- only when $B^{(0)}(\vec{x}) \geq 0$



Contributons

- Barrier Functions $\rightarrow$ CBFs; sampling/fixing control inputs
- Multiple Barrier Functions
- Successive Barrier Functions
- Runtime enforcement
  - Dwell time bounds
  - Monitor Synthesis
- Experimental Evaluations

Synthesis of Multiple and Successive CBFs

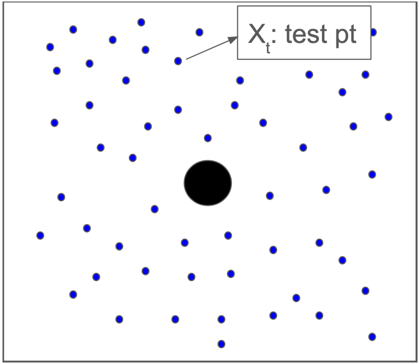
Barrier Synthesis using SOS

$$\text{Find} \quad B(\vec{x}) \ s.t.$$

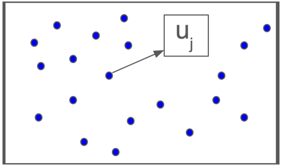
$$\left.\begin{aligned} \forall \vec{x} \in X_u, B(\vec{x}) > 0 \\ \forall \vec{x} \in X_o, B(\vec{x}) < 0 \\ \forall \vec{x}, \nabla B(\vec{x}) \cdot f(\vec{x}) \leq -\lambda B(\vec{x}) \end{aligned}\right\}$$

Enforced using SOS
+
Putinar's Positivstellensatz
[Parillo et al.]

Synthesis of Multiple Barriers

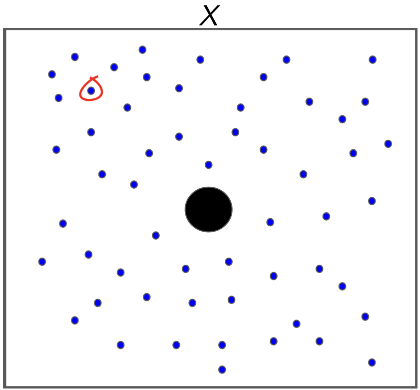


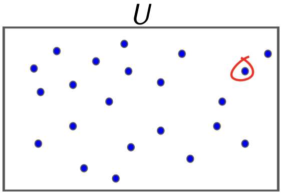
$x_t \in X$



$u_j \in U_{fin}$

Synthesis of Multiple Barriers





Synthesize a Barrier B_j:

- Fix $u = u_j \in U_{fin}$,
- $x_0 = x_t$

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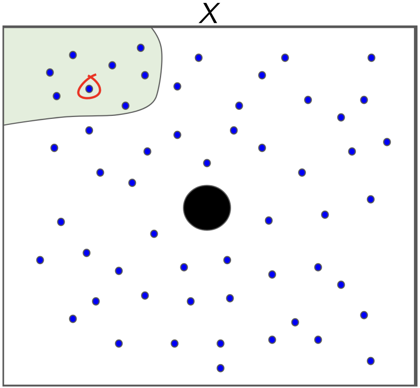

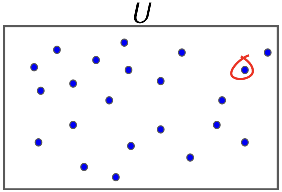

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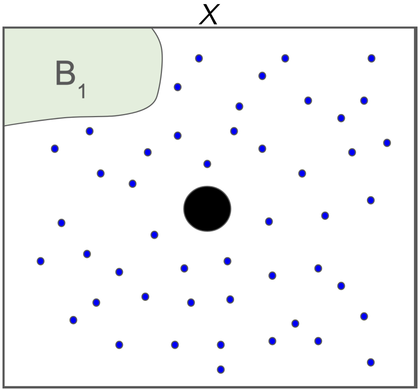

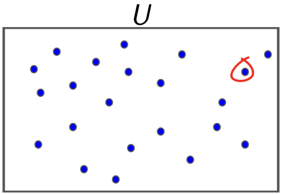

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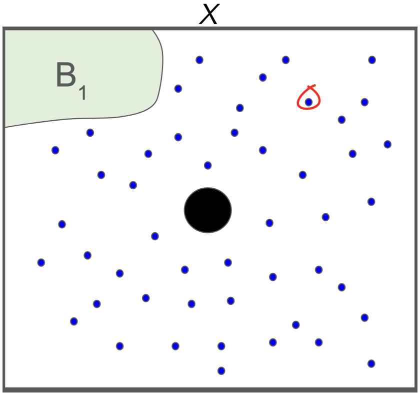

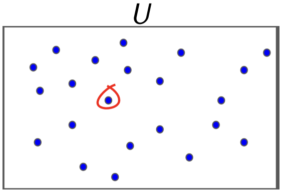

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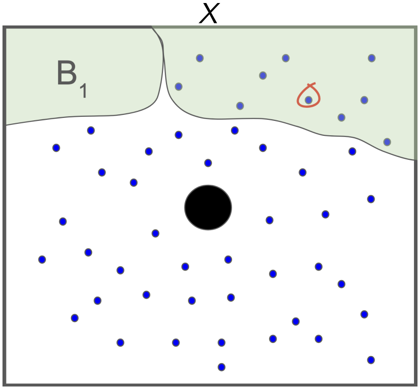

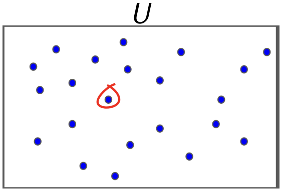

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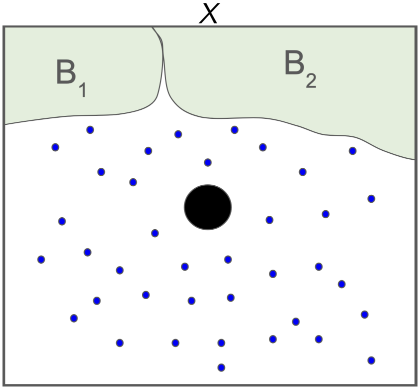

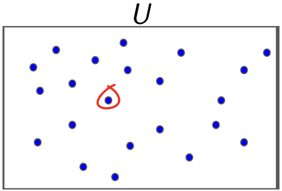

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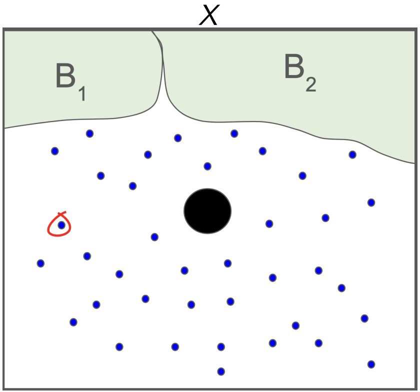

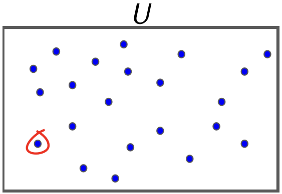

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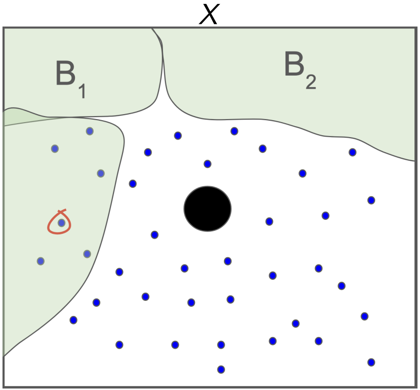

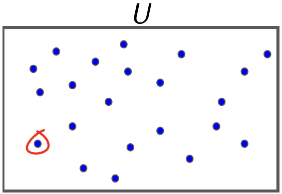

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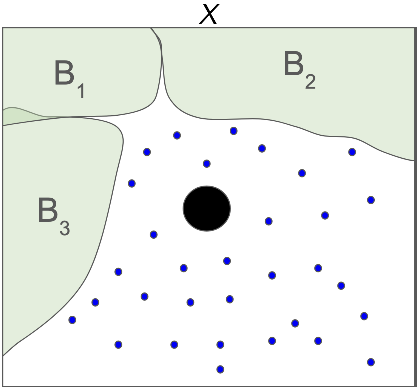

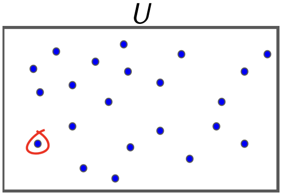

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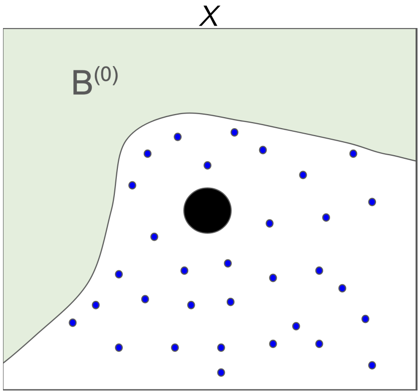

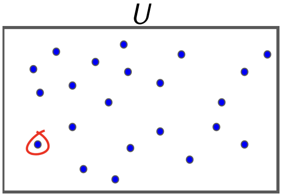

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- Fix $u = u_j \in U_{fin}$,
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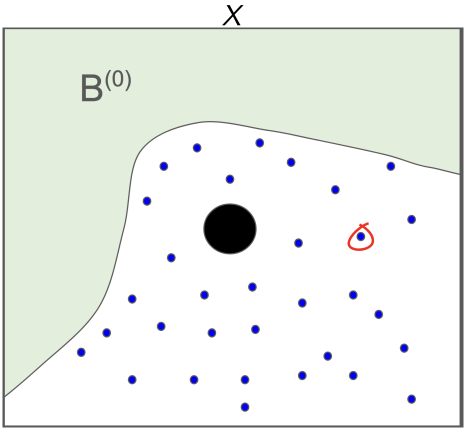
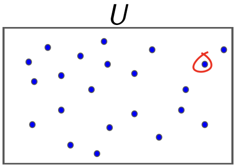
$$B^{(0)} = \min(B_1, \ldots, B_k)$$

Synthesis of Successive Barriers





Synthesize a Successive Barrier B_j^k:

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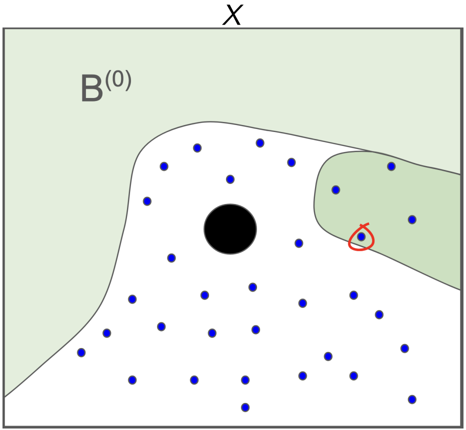

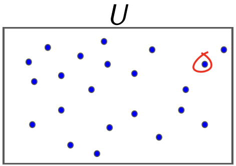

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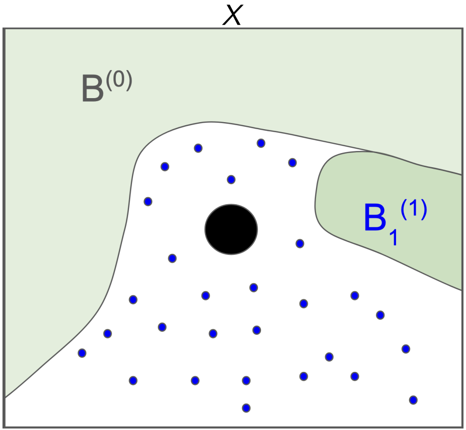

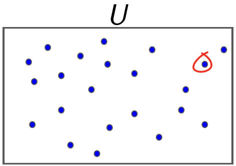

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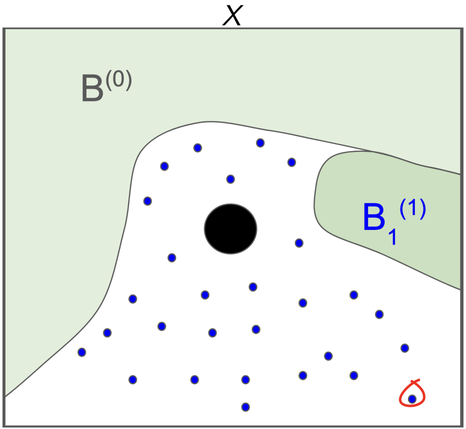

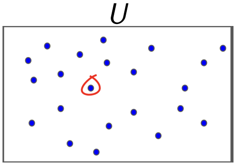

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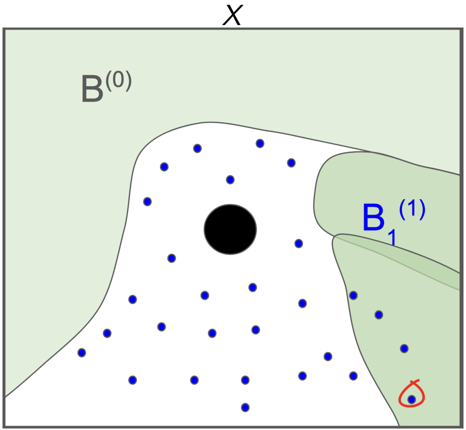

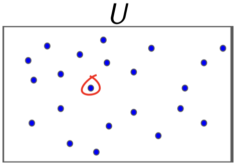

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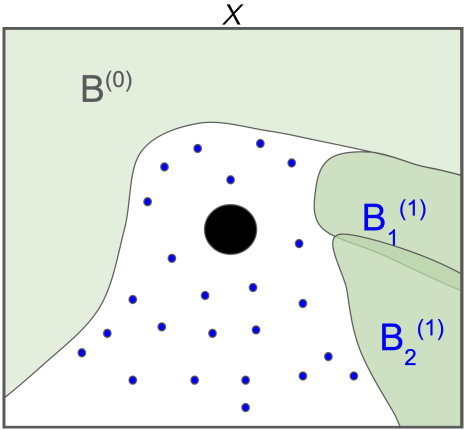

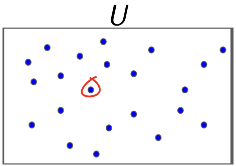

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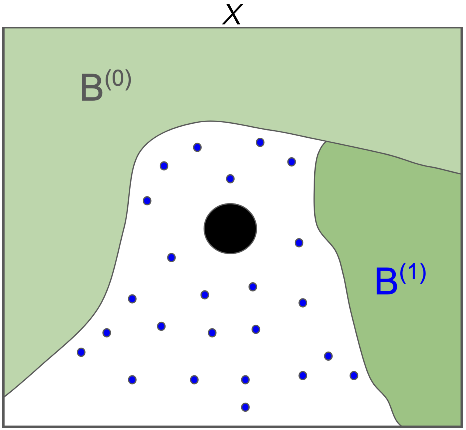

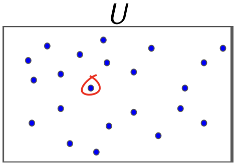

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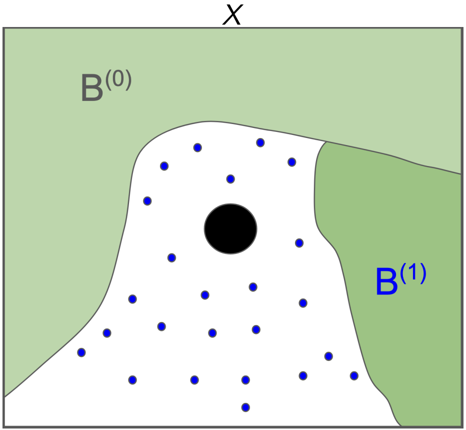
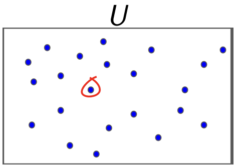
$$B^{(0)} = \min(B_1^{(0)}, \ldots, B_k^{(0)}),\ B^{(1)} = \min(B_1^{(1)}, \ldots, B_k^{(1)})$$

Synthesis of Successive Barriers





Synthesize a Successive Barrier B_j^k:

- Fix $u = u_j \in U_{fin}$, $x_0 = x_t$
- $B^{(0)}(\vec{x}) \geq 0 \wedge B^{(1)}(\vec{x}) \geq 0 \implies \nabla B_j \cdot f(\vec{x}, \vec{u_j}) \leq -\lambda B_j(\vec{x})$,

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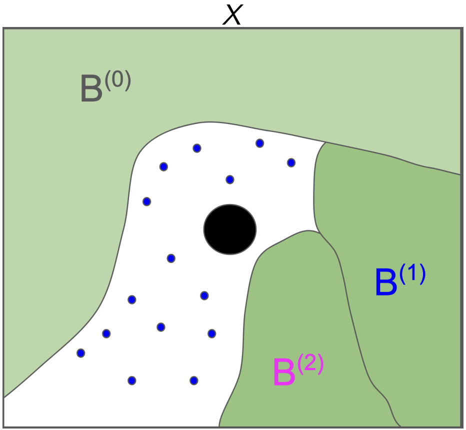
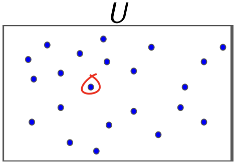
Synthesis of Successive Barriers





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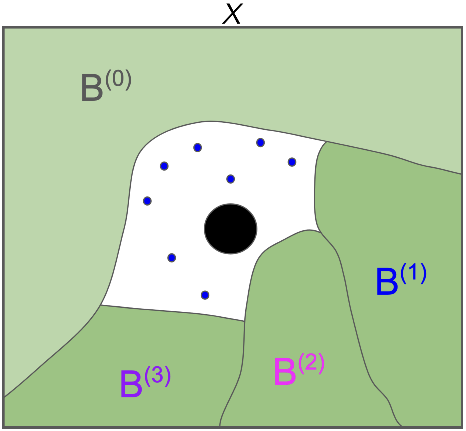
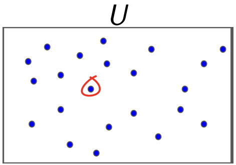
Synthesis of Successive Barriers





Synthesize a Successive Barrier B_j^k:

- Fix $u = u_j \in U_{fin}$, $x_0 = x_t$
- $B^{(0)}(\vec{x}) \geq 0 \wedge B^{(1)}(\vec{x}) \geq 0$
  $\wedge B^{(2)}(\vec{x}) \geq 0 \implies$

$\nabla B_j \cdot f(\vec{x}, \vec{u}_j) \leq -\lambda B_j(\vec{x})$,

$$B^{(0)} = \min(B_1^{(0)}, \ldots, B_k^{(0)}),\ B^{(1)} = \min(B_1^{(1)}, \ldots, B_k^{(1)}),\ B^{(2)} = \min(B_1^{(2)}, \ldots, B_k^{(2)})$$

Monitoring

Monitoring

Successive Barriers

$$B^{(0)}_{min}(\vec{x}) = \min(B^{(0)}_1(\vec{x}), \ldots, B^{(0)}_m(\vec{x}))$$

$$\Big\downarrow \; B^{(0)}_{min}(\vec{x}) \geq 0$$

$$B^{(1)}_{min}(\vec{x}) = \min(B^{(1)}_1(\vec{x}), \ldots, B^{(1)}_m(\vec{x}))$$

$$\Big\downarrow \; B^{(1)}_{min}(\vec{x}) \geq 0$$

$$\vdots$$

$$\Big\downarrow \; B^{(k-1)}_{min}(\vec{x}) \geq 0$$

$$B^{(k)}_{min}(\vec{x}) = \min(B^{(k)}_1(\vec{x}), \ldots, B^{(k)}_m(\vec{x}))$$

Monitoring

Successive Barriers

$$B^{(0)}_{min}(\vec{x}) = \min(B^{(0)}_1(\vec{x}), \ldots, B^{(0)}_m(\vec{x}))$$

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$$\Big\downarrow \; B^{(1)}_{min}(\vec{x}) \geq 0$$

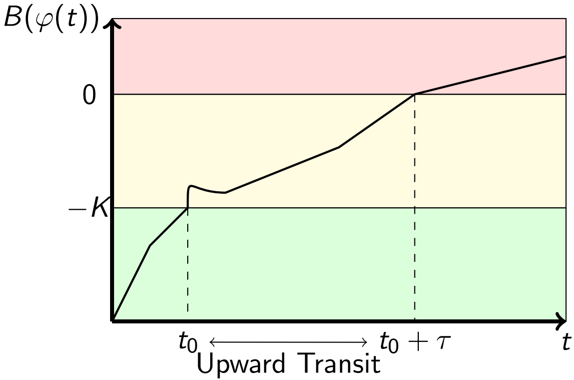
$$\vdots$$

$$\Big\downarrow \; B^{(k-1)}_{min}(\vec{x}) \geq 0$$

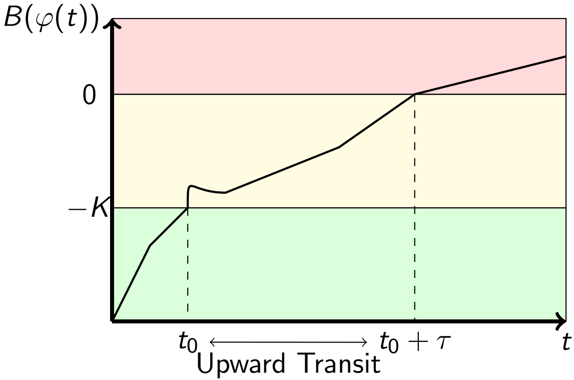
$$B^{(k)}_{min}(\vec{x}) = \min(B^{(k)}_1(\vec{x}), \ldots, B^{(k)}_m(\vec{x}))$$

- Monitoring using QP [Ames et al.]
- QP is solved in "continuous time"
- Dwell time bounds

Monitor Synthesis

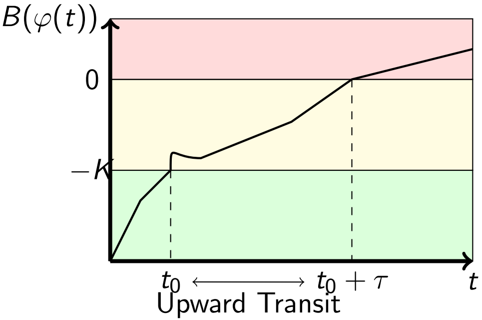


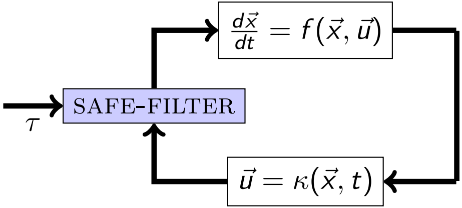
Monitor Synthesis



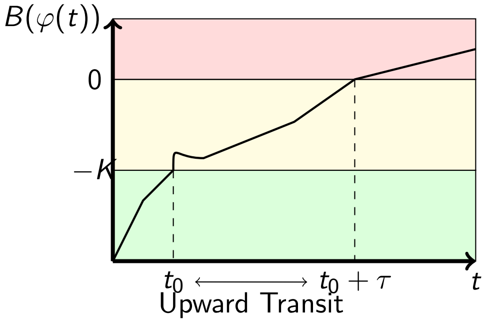
- unsafe region; $B(\vec{x}) \geq 0$
- transit region; $B(\vec{x}) \in [-K, 0)$
- safe region; $B(\vec{x}) < -K$

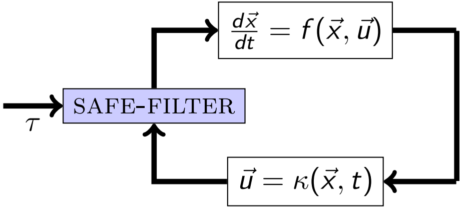
Monitor Synthesis





Monitor Synthesis





$$\text{SAFE-FILTER}(\vec{x}; B_{min}) = \begin{cases} \textsc{Pass} & \text{if } B_{min}(\vec{x}) < -K \\ \textsc{Override}(\vec{u}_{min}(\vec{x})) & \text{otherwise} \end{cases}$$

Monitor Synthesis

Successive Barriers

$$B_{min}^{(0)}(\vec{x}) = \min(B_1^{(0)}(\vec{x}), \ldots, B_m^{(0)}(\vec{x}))$$

$$\Big\downarrow \; B_{min}^{(0)}(\vec{x}) \geq 0$$

$$B_{min}^{(1)}(\vec{x}) = \min(B_1^{(1)}(\vec{x}), \ldots, B_m^{(1)}(\vec{x}))$$

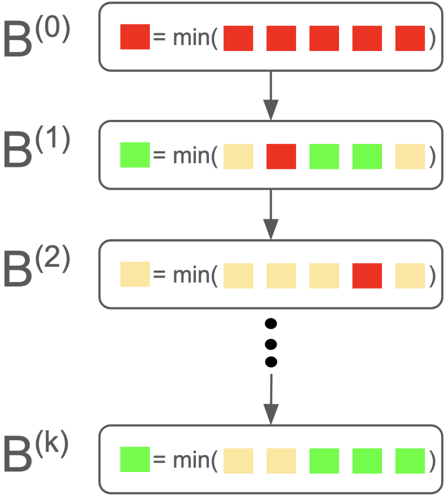
$$\Big\downarrow \; B_{min}^{(1)}(\vec{x}) \geq 0$$

$$\vdots$$

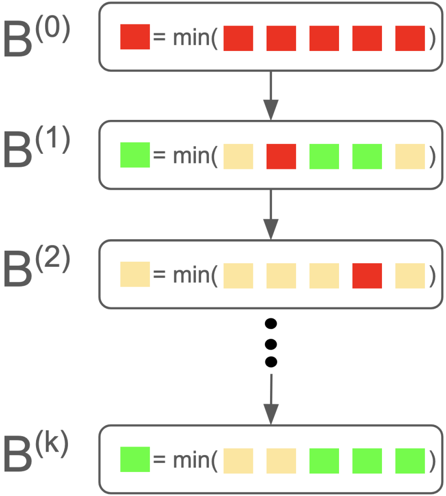
$$\Big\downarrow \; B_{min}^{(k-1)}(\vec{x}) \geq 0$$

$$B_{min}^{(k)}(\vec{x}) = \min(B_1^{(k)}(\vec{x}), \ldots, B_m^{(k)}(\vec{x}))$$

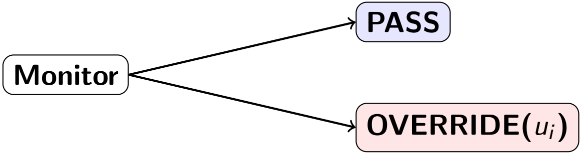
Monitor Synthesis



Monitor Synthesis



→ Synthesizing a finite state machine:



Evaluations

Evaluations - 2D

2D Nonlinear Dynamics with 2 control inputs $u_1, u_2 \in [-0.1, 0.1]^2$;

$$\dot{x}_1 = \frac{1}{2}x_1 - \frac{1}{5}x_2 - \frac{1}{100}x_1x_2 - \frac{1}{2}u_1 + \frac{1}{2}u_2,$$
$$\dot{x}_2 = x_1 - \frac{2}{5}x_2 - \frac{1}{20}x_2^2 - \frac{7}{10}u_2 + \frac{1}{10}u_1,$$

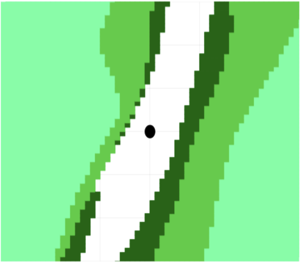


Figure: Improvement in CI region with three iterations of successive barrier functions

Evaluations - 4D

4D Nonlinear Dynamics with 2 control inputs $u_1, u_2 \in [-0.1, 0.1]^2$;

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{1}{2}x_1 - \frac{1}{5}x_2 + \frac{1}{20}x_3x_1 - \frac{1}{100}x_1x_2 - \frac{1}{2}u_1, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -\frac{2}{5}x_4 + \frac{1}{5}x_1 - \frac{1}{20}x_3^2 - \frac{7}{10}u_2,
\end{aligned}
$$

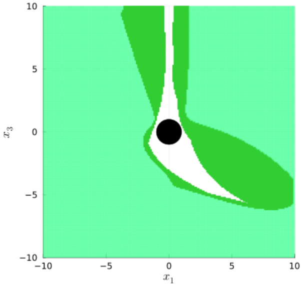


Figure: Improvement in CI region with two iterations of successive barrier functions

Evaluations - 5D

5D coordinated turn model with 2 control inputs $u_1, u_2 \in [-5, 5]^2$;

$$\begin{aligned}
\dot{x}_1 &= x_3 \cos x_4, \\
\dot{x}_2 &= x_3 \sin x_4, \\
\dot{x}_3 &= u_1, \\
\dot{x}_4 &= x_5, \\
\dot{x}_5 &= u_2,
\end{aligned}$$

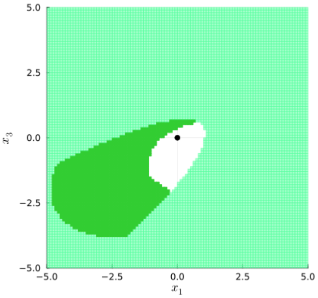


Figure: Improvement in CI region with two iterations of successive barrier functions

Comparison with FOSSIL

			FOSSIL		Ours		
system	dim	inputs	success	time(s)	success	time(s): $B^{(1)}$	# barriers
poly1	2	1	✓	3.2	✓	**0.6**	2
poly2	2	1	✓	1.9	✓	**1.0**	2
van der Pol	2	1	✓	4.8	✓	**2.0**	2
inv pendulum	2	1	✓	3.2	✓	**2.0**	2
poly3	2	2	✓	**1.3**	✓	7.6	4
poly4	3	2	✓	74.8	✓	**6.8**	4
poly5	4	2	✗	-	✓	**36.4**	4
coord turn	5	2	✗	-	✓	**108.4**	4
planar multirotor	6	2	✗	-	✓	**131.6**	4

Comparison with FOSSIL

system	dimensions	inputs	F	S	$\neg F \wedge S$	F $\wedge \neg S$
poly1	2	1	417	**649**	**375**	143
poly2	2	1	716	**922**	**215**	9
van der Pol	2	1	507	**649**	**322**	180
inv pendulum	2	1	683	**798**	**193**	78
poly3	2	2	556	**977**	**421**	0

Comparison with FOSSIL

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inv pendulum	2	1	683	**798**	**193**	78
poly3	2	2	556	**977**	**421**	0

Contributons

- Barrier Functions $\rightarrow$ CBFs; sampling/fixing control inputs
- Multiple Barrier Functions
- Successive Barrier Functions
- Runtime enforcement
  - Dwell time bounds
  - Monitor Synthesis
- Experimental Evaluations

The implementation and resulting barrier functions are available at
`https://github.com/rameezw/SuccessiveBarriers`

Thank You